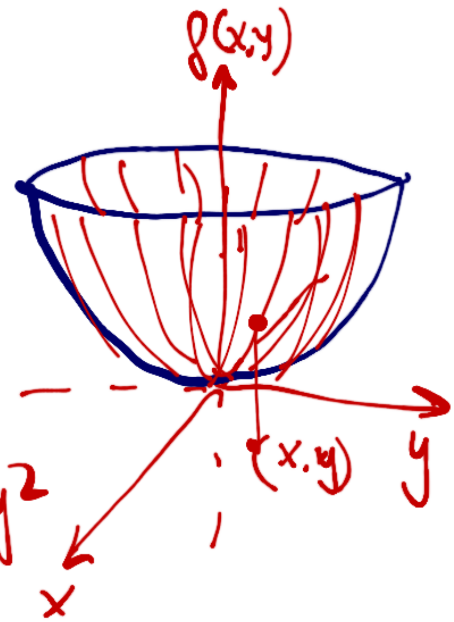
Summary Differentiability in $\mathbb{R}^n$

A function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ assigns any arbitrary point in $\mathbb{R}^2$ to just one value in $\mathbb{R}$

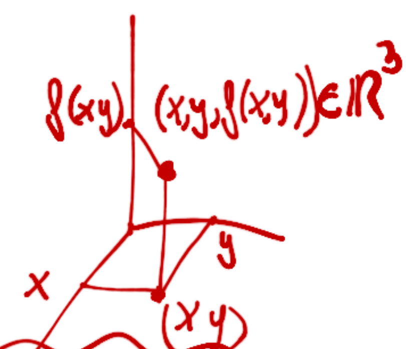
$$(x,y) \longrightarrow f(x,y) \in \mathbb{R}$$



Example: $f(x,y) = x^2 + y^2$

$$(x,y) \longrightarrow f(x,y) = x^2 + y^2$$

Its graph is a surface !!

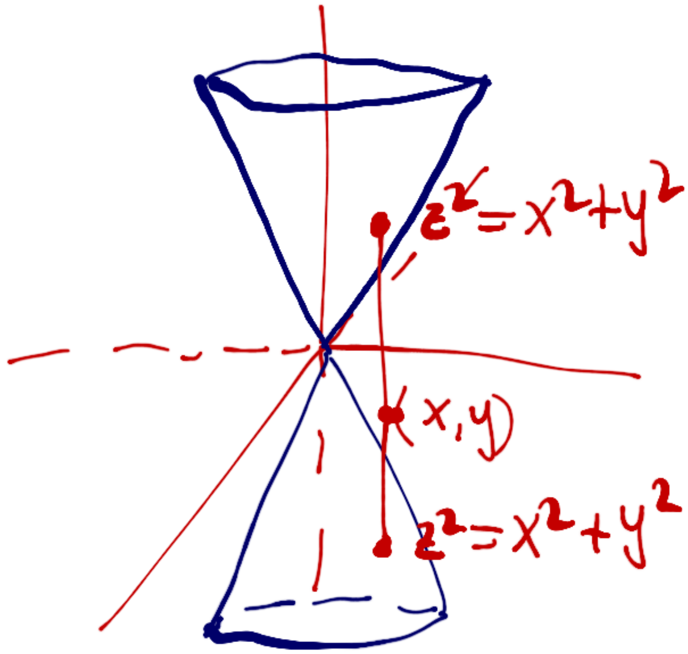


* Such a surface can be written by a formula

$$z = f(x,y)$$

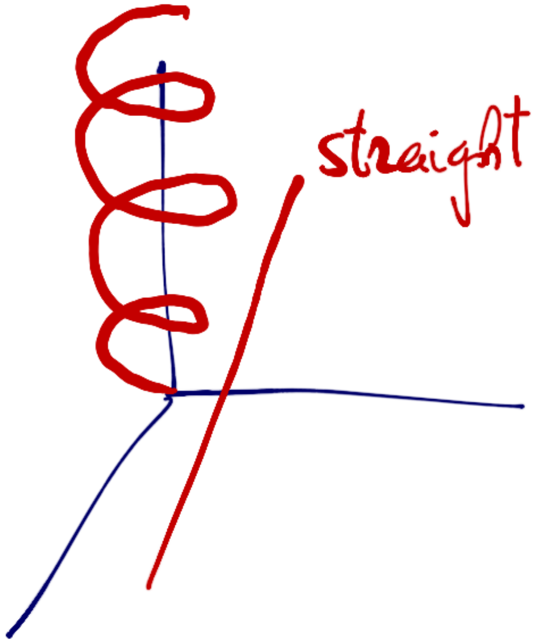
However, that surface might not be a function.

$z^2 = x^2 + y^2$ Cone



Different functions:

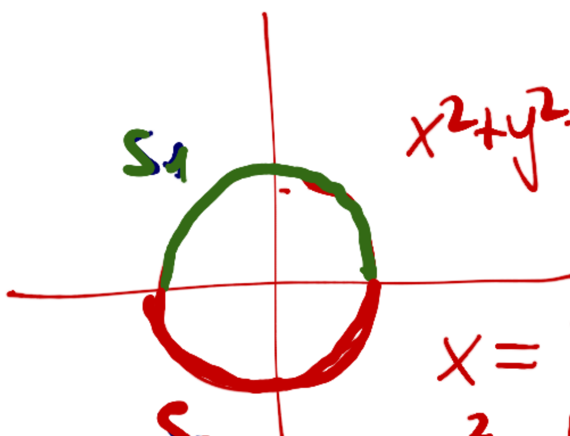
- Scalar functions : $f : \mathbb{R}^n \to \mathbb{R}$
  (magnitudes)
- Vector functions : $F : \mathbb{R}^n \to \mathbb{R}^m$
  (velocity, flux, etc)
- Trajectories $s : \mathbb{R} \to \mathbb{R}^3$
  $t \to s(t) = (s_1(t), s_2(t), s_3(t))$



straight line.

$s(t) \equiv$ parametric expression.

$s(t) \in \mathbb{R}$ given by a parameter t.



$x^2+y^2=1$

$$\left.\begin{aligned} x &= t \\ y^2 &= 1-t^2 \end{aligned}\right\}$$

$S_1(t) = \left(t, \sqrt{1-t^2}\right) \qquad S_1 : \mathbb{R} \to \mathbb{R}^2$

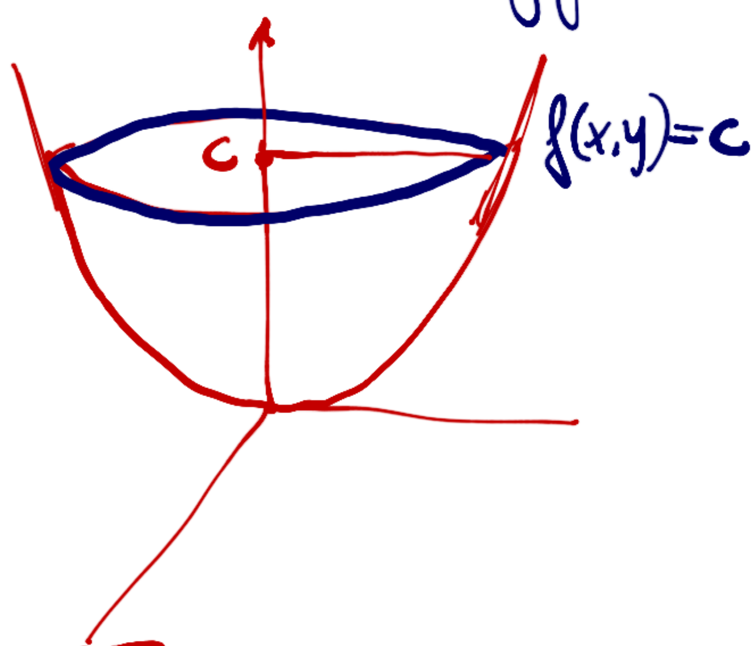
$S_2(t) = \left(t, -\sqrt{1-t^2}\right) \qquad S_2 : \mathbb{R} \to \mathbb{R}^2$

Level curves (or level surfaces)

Curves where the function is constant

$$f(x,y) = C \,, \quad C \in \mathbb{R}.$$

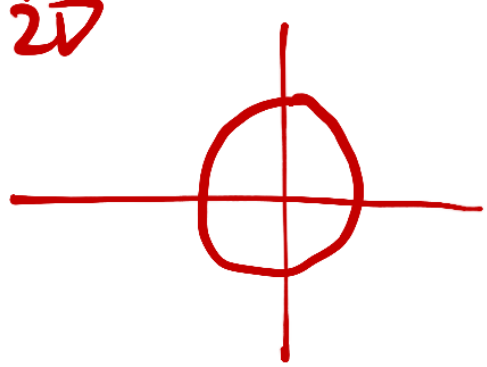
They allow us to draw 3D figures into 2D.



$$f(x,y) = x^2 + y^2$$

Level curve: $\underline{x^2+y^2=c}$ in 2D

<u>Circumferences</u>

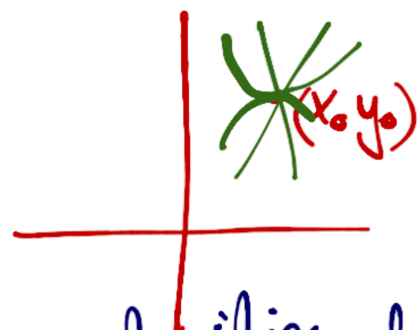


Limits and continuity

If the limit exists it must be unique

This means it **CANNOT DEPEND** on the trajectory.

In 2D



- Approaching following families of functions
  $y = \lambda x$
  $y = \lambda x^2$
  around the point
  around $(0,0)$

  If we get a value **DOES NOT IMPLY** that the limit exists

- Polar coordinates. We get a limit if does not depend on θ.

Continuity at x_0 : $\lim_{x \to x_0} f(x) = f(x_0) \in \mathbb{R}$.

$x_0 \in \mathbb{R}^N$

Differentiability at x_0 :

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

direction.

Derivatives : Partial derivatives.

definition
$$\begin{cases} \dfrac{\partial f(x_0 y_0)}{\partial x} = \lim_{h \to 0} \dfrac{f((x_0,y_0)+h(1,0)) - f(x_0,y_0)}{h} \\[2ex] \dfrac{\partial f(x_0 y_0)}{\partial y} = \lim_{k \to 0} \dfrac{f((x_0 y_0)+k(0,1)) - f(x_0 y_0)}{k} \end{cases}$$

direction of y

$$f((x_0 y_0)+h(1,0)) = f(x_0+h, y_0)$$

If $(x_0,y_0)=(0,0)$, $f(h,0)$

values for derivatives.

ratio of change in the direction of the axis. (vector of standard basis.)

Directional derivative. $D_v f$

ratio of change in the direction v

* Existence of derivatives

DOES NOT IMPLY diff..

DOES NOT IMPLY cont.

* Differentiability means existence of tangent plane (smooth function)

Analytically: Existence of partial derivatives

$$+$$

$$0 = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}{\|(x,y)-(x_0,y_0)\|}$$

$f: \mathbb{R}^2 \to \mathbb{R}$.

or

Existence + continuity of partial derivatives

$$\Downarrow$$

differentiability

$$\Downarrow$$

$$\lim \frac{\left\| F(x,y) - F(x_0,y_0) - JF(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \right\|}{\|(x,y)-(x_0,y_0)\|} = 0$$

$F: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad JF \equiv 2\times 2$ matrix

In $\mathbb{R}^3$, $f: \mathbb{R}^3 \to \mathbb{R}$

$$\frac{\overline{f(x,y,z) - f(x_0, y_0, z_0) - \frac{\partial f}{\partial x}(x-x_0) - \frac{\partial f}{\partial y}(y-y_0) - \frac{\partial f}{\partial z}(z-z_0)}}{\|(x,y,z) - (x_0, y_0, z_0)\|}$$

$$Df = \nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

<u>If f is diff.</u> / f a function

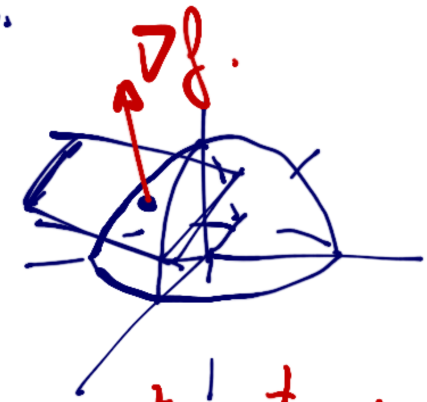
$$D_v f(x_0) = \langle \underbrace{\nabla f(x_0)}_{\text{gradient}}, v \rangle \quad \text{with } \|v\| = 1$$

Tangent plane:
$$z = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)$$

- Tangent plane to a surface.

$$\boxed{f(x,y,z)=0}$$

Implicit form.



gradient vector is orthogonal to the surface

Tangent plane:

$$\underbrace{\nabla f(x_0,y_0,z_0)\cdot\begin{pmatrix}x-x_0\\ y-y_0\\ z-z_0\end{pmatrix}=0}$$

Example: $z^2=x^2+y^2$ Cone $\sim f(x,y,z)=x^2+y^2-z^2 \underset{\shortparallel}{} 0$

Tangent plane at $(1,1,1)$

$$\nabla f(x,y,z)=(2x,2y,-2z)$$

$$\nabla f(1,1,1)=(2,2,-2)\Rightarrow$$

Tangent plane.

$$\underline{(2,2,-2)}\cdot\begin{pmatrix}x-1\\ y-1\\ z-1\end{pmatrix}=0$$

- $f(x,y) = x^2+y^2$ function.

Equivalent to $\underbrace{z = x^2+y^2}_{\text{Formula.}} \equiv$ surface.

$$g(x,y,z) = x^2+y^2-z.$$

$$\nabla g(x,y,z) = (2x, 2y, -1)$$

At $(1,1,2)$ on the surface.

$$f(1,1) = 2 \qquad \nabla f = (2x, 2y) \Rightarrow \nabla f(1,1) = (2,2)$$

option a) : $z = f(1,1) + \nabla f(1,1) \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$

$$z = 2 + 2(x-1) + 2(y-1)$$

$$2(x-1) + 2(y-1) - (z-2) = 0$$

option b) : $\nabla g(1,1,2) \begin{pmatrix} x-1 \\ y-1 \\ z-2 \end{pmatrix} = (2,2,-1) \begin{pmatrix} x-1 \\ y-1 \\ z-2 \end{pmatrix} = 0$

Chain rule:

$$D(g\circ f)(x_0) = Dg(f(x_0)) \cdot Df(x_0)$$

matrices.

- In 1D $y=f(x)$ and $x=x(t)$

then $y = f(x(t)) = h(t)$

So $$\frac{dy}{dt} = \underbrace{\frac{dy}{dx}}_{\frac{df(x(t))}{dt}} \frac{dx}{dt} = \frac{df(x(t))}{dx} \cdot \frac{dx}{dt}$$

In 2D $\underline{z = f(x,y)}$, $x \equiv x(t)$

$y \equiv y(t)$

$\underline{f : \mathbb{R}^2 \longrightarrow \mathbb{R}}$. $\mathcal{D}f = \nabla f$

$$z = f\big(x(t), y(t)\big) = f\big(s(t)\big)$$

$$s(t) = \big(x(t), y(t)\big)$$

$$s : \mathbb{R} \longrightarrow \mathbb{R}^2$$

Thanks to the chain rule

$$\frac{dz}{dt} = \frac{df(s(t))}{ds} \cdot \frac{ds}{dt} = \nabla f(s(t)) \cdot \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

$$= \underbrace{\left(\frac{\partial z}{\partial x} , \frac{\partial z}{\partial y} \right)}_{\nabla f} \cdot \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

Example: $\boxed{z = x^2 y + 3xy^4}$

$$\left.\begin{array}{l} x = \sin 2t \\ y = \cos t \end{array}\right\} \text{ Find } \frac{dz}{dt}$$

$$z = x^2(t)\,y(t) + 3\,x(t)\,y^4(t)$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

$$= \underbrace{\left(2xy + 3y^4\right)}_{\frac{\partial z}{\partial x}}\ \underbrace{2\cos 2t}_{\frac{dx}{dt}} - \underbrace{\left(x^2 + 12xy^3\right)}_{\frac{\partial z}{\partial y}}\underbrace{\sin t}_{\frac{\partial y}{\partial t}}$$

$$\frac{dz}{dt} = 2\,x(t)\frac{dx}{dt}\,y(t) + x^2(t)\frac{dy}{dt}$$

$$+ 3\,y^4(t)\frac{dx}{dt} + 12\,x(t)\,y^3(t)\frac{dy}{dt}$$

Problem 10 in $\boxed{1.3}$

ii) $s'(t) = F(s(t))$

$s(t) = \left(e^{2t}, \frac{1}{t}, \log t\right)$, $\underline{F(x,y,z)} = (\underline{2x}, \underline{-y^2}, \underline{y})$

$\Downarrow$

$$s'(t) = \frac{ds(t)}{dt} = \left(2e^{2t}, -\frac{1}{t^2}, \frac{1}{t}\right)$$

$$F(s(t)) = F\left(\underline{e^{2t}}, \underline{\frac{1}{t}}, \log t\right) =$$

$$= \left(2e^{2t}, -\frac{1}{t^2}, \frac{1}{t}\right)$$

- The pressure P (in kilopascals) volume V (liters) and temperature T (in Kelvins) of a mole of an ideal gas are related by the equation

$$\boxed{PV = 8'31\ T}$$

Find the rate at which pressure is changing when T = 300 K and increasing at a rate of 0'1 K/s and V = 100 L and increasing at a rate of 0'2 L/s

$$P = \frac{8'31\ T}{V} = P(T, V) \equiv P(T(t), V(t))$$

$T = 300K \quad \frac{dT}{dt} = 0'1, \quad V = 100 \quad \frac{dV}{dt} = 0'2$

rate of change for $P \equiv \frac{dP}{dt}$, $P = \frac{8'31\,T}{V}$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T} \underbrace{\frac{dT}{dt}}_{0'1} + \frac{\partial P}{\partial V} \underbrace{\frac{dV}{dt}}_{0'2}$$

$$= \frac{8'31}{V} \cdot 0'1 - \frac{8'31\,T}{V^2} \cdot 0'2$$

$$= \frac{8'31}{100} \cdot 0'1 - \frac{8'31 \cdot 300}{(100)^2} \cdot 0'2 = \underline{-0'04155}$$

The pressure decreases at that rate.

Implicit differentiation

Write a function

Implicit form	Explicit form	Derivative
$xy = 1$	$y = \frac{1}{x} \Rightarrow$	$\frac{dy}{dx} = \frac{-1}{x^2}$

However we cannot use those two forms when we are unable to solve the equation for y as a function of x

$$\boxed{x^2 - 2y^3 + 4y = 2}$$

To differentiate we must use

Implicit differentiation (Chain rule)

- Every term depending on x we differentiate as usual
- Every term involving $y \equiv y(x)$ we must apply the chain rule.

Example: $x^2+y^2=1$, $y \equiv y(x)$

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[1]=0$$

$$2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy(x)}{dx}=\frac{-x}{y}$$

In several variables.

$$F(x,y)=0 \quad \text{or} \quad F(x,y(x))=0$$

Applying the chain rule. $\displaystyle \frac{dF}{dx}=\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}=0$

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}}$$

Ex: $x^3 + y^3 = 3xy \Rightarrow F(x,y) = x^3 + y^3 - 3xy$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{3x^2 - 3y}{3y^2 - 3x} = -\frac{x^2 - y}{y^2 - x}$$

explict form

If $\boxed{z = f(x,y)}$ in implicit form

$F(x, y, f(x,y)) = 0$

$f : \mathbb{R}^2 \to \mathbb{R}$

(x,y) indep.t variables.

Using the chain rule.

$$\frac{\partial F}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y} \underbrace{\frac{dy}{dx}}_{=0} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

Infectious disease outbreak size.

Mathematical models predict the fraction of population that will be infected when a disease begins to spread.

$$\underbrace{\rho e^{-q\Delta} - 1 + \Delta}_{F(\rho, q, \Delta)} = 0 \qquad \boxed{\rho e^{-q\Delta} = 1 - \Delta}$$

Kermack-McKendrick model

$\Delta \equiv$ fraction of population infected.

$\rho \equiv$ initially population susceptible to infection

$q \equiv$ measure of transmissibility

How does the outbreak size Δ change with an increase in q?

$$F(\rho, q, \Delta) = \rho e^{-q\Delta} - 1 + \Delta$$

$$\boxed{\frac{\partial \Delta}{\partial q} = - \frac{\partial F / \partial q}{\partial F / \partial \Delta} = \frac{\square}{\square}}$$

$$F\left(\ell, q, \Delta(q)\right) = 0$$

$$\underbrace{\frac{\partial F}{\partial \ell} \cdot \underbrace{\frac{d\ell}{dq}}_{=0}}_{=0} + \frac{\partial F}{\partial q} \underbrace{\frac{dq}{dq}}_{=1} + \frac{\partial F}{\partial \Delta} \frac{\partial \Delta(q)}{\partial q} = 0$$

$$\frac{\partial \Delta}{\partial q} = - \frac{\partial F / \partial q}{\partial F / \partial \Delta}$$

$$F\left(\ell, q, \Delta(q)\right) = \boxed{\ell e^{-q/\Delta(q)} - 1 + \Delta(q) = 0}$$